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Elastic and sound orthonormal beams and localized fields in linear mediums: I. Basic equations

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Abstract

Basic definitions and general relations for elastic and sound fields, defined by a given set of orthonormal functions on a real manifold, are presented. The proposed mathematical formalism makes it possible to obtain families of orthonormal beams and localized fields in both isotropic and anisotropic linear elastic mediums as well as the similar sound fields in an ideal liquid. All these fields are described as superpositions of plane waves whose intensities and phases are specified by a set of orthonormal scalar functions on a two- or three-dimensional manifold. By way of illustration, the fields defined by the spherical harmonics are considered.

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1. Introduction

In recent decades, electromagnetic, elastic and sound beams have been studied extensively and many interesting solutions of the wave equation, such as fractional solutions [1], non-diffracting—Bessel and Bessel–Gauss—beams [2], and various localized fields (see, for example, [3–12] and references therein), have been suggested. At the beginning of the 1980s, Brittingham [3] proposed the problem of searching for specific electromagnetic waves—focus wave modes (FWMs)—having a three-dimensional pulse structure, being non-dispersive for all time, and moving at light velocity in straight lines. A number of packet-like solutions have been presented [3–7]. Although the FWM has an infinite total energy, a superposition of the FWMs can produce finite-energy pulses that exhibit extended ranges of localization [4–7]. It was shown by Ziolkowski *et al* [5] that such pulses can be excited from finite apertures. In 1985, Wu [8] introduced a conception of electromagnetic missiles moving at light velocity and having a very slow rate of decrease with distance. Recently, Kiselev and Petel [9] presented packet-like solutions with Gaussian localization in both longitudinal and transverse directions.

The Fourier transform and plane-wave expansions play a very important role in the analysis of localized fields. By using the Fourier transform, a method for obtaining separable and non-separable, localized solutions of constant coefficient homogeneous partial differential equations was developed by Donnelly and Ziolkowski [6]. In the frame of this approach, the evanescent fields and the causality of the FWMs have been studied comprehensively elsewhere [7]. In particular, it was shown that the source-free FWMs are composed solely of backward- and forward-propagating homogeneous plane waves.

By using expansions in plane waves, we have introduced [10–12] a specific type of linear field—beams defined by a set of orthonormal scalar functions on a two-dimensional or three-dimensional manifold (beam manifold $\mathcal{B}$). The proposed approach enables one to obtain a set of orthonormal beams, normalized to either the energy flux through a given plane (beams with two-dimensional $\mathcal{B}$) or the total energy transmitted through this plane (beams with three-dimensional $\mathcal{B}$), and various families of localized fields: three-dimensional standing waves, moving and evolving whirls, etc. This can be applied to any linear field, such as electromagnetic waves in free space, isotropic, anisotropic and bianisotropic mediums, elastic waves in isotropic and anisotropic mediums, sound waves, weak gravitational waves, etc.

On this basis, we have found [10–12] unique families of exact solutions of the homogeneous Maxwell equations for electromagnetic waves in isotropic mediums and free space. The families of orthonormal beams [10, 11] form convenient functional bases for more complex electromagnetic fields. This provides a means to generalize the free-space techniques [13] for characterizing complex mediums and the covariant wave-splitting technique [14] to the case of incident beams. The families of localized electromagnetic fields (electromagnetic storms, whirls and tornadoes) [10–12] also possess very interesting properties. A small (about several wavelengths) clearly defined core region with maximum intensity of field oscillations is an inherent feature of these fields. For an electromagnetic storm, the time average energy flux vector $\boldsymbol{S}$ is identically zero at all points. For both whirls and tornadoes, the radial S_{R}, the azimuthal S_{A}, and the normal (to a given plane) S_{N} cylindrical components of $\boldsymbol{S}$ are independent of the azimuthal angle ψ, besides $S_{\mathrm{R}} = S_{\mathrm{N}} \equiv 0$ for whirls. As a result, whirls and tornadoes have circular and spiral energy flux lines, respectively. The solutions, which describe electromagnetic whirls moving without dispersion with speed $0 < V < c$, finite-energy evolving electromagnetic whirls [10, 11], and weak gravitational orthonormal beams and localized fields [12], have been found as well.

The mathematical formalism proposed in [11] can be applied to scalar, vector and tensor plane-wave superpositions. However, the special features of its application to obtain sets of exact solutions to one or another of the wave equations, as well as the properties of the resulting solutions, substantially depend on the properties of the partial eigenwaves. The corresponding mathematical framework for two types of transverse waves—vector (electromagnetic) and tensor (weak gravitational)—has been presented in [11, 12]. In this paper, to treat sound fields in liquids and elastic fields in solids, this framework is augmented to include also scalar, vector and tensor plane waves describing the sound pressure fields in liquids, the displacement vector fields for longitudinal (quasi-longitudinal) and transverse (quasi-transverse) waves in isotropic (anisotropic) solids, and the corresponding deformation and stress tensor fields.

The objectives of the current series of papers are twofold:

(1) We extend the proposed formalism to the cases of elastic waves in isotropic and anisotropic mediums and sound waves in an ideal liquid.

(2) On this basis, we obtain unique solutions which describe orthonormal beams and localized fields in an isotropic elastic medium and an ideal liquid. They illustrate both similarities of and distinctions between scalar (sound waves), vector (radial displacement vector for longitudinal elastic waves, meridional and azimuthal displacement vectors for transverse elastic waves) and tensor (deformation and stress tensors) plane-wave superpositions defined by the same set of functions, such as the spherical harmonics.

Accordingly, in this paper basic definitions and general relations for the plane-wave superpositions, defined by a given set of orthonormal functions on a real manifold, are presented in section 2. In section 3, we discuss the properties of elastic and sound harmonic plane waves (eigenwaves) and supply the relations that are necessary to apply the general theory to both isotropic and anisotropic mediums. A mathematical formalism convenient for analysis of orthonormal beams and localized fields is briefly outlined in section 4. Three subsequent papers will deal with the superpositions of longitudinal (paper II) and transverse (paper III) elastic plane waves in an isotropic medium, and sound plane waves in an ideal liquid (paper IV).

2. Fields defined by a set of orthonormal functions on a real manifold

To compose a field from eigenwaves in a linear medium, we must specify all eigenwave properties: propagation directions, frequencies or wave numbers, polarizations, intensities and phases. Of course, in the case of elastic waves in an anisotropic medium, eigenwave polarizations are specified by the medium itself, and we have to set just propagation directions, intensities and phases.

The fields (beams) defined by a set of orthonormal functions (u_n) on a two- or three-dimensional real manifold $\mathcal{B}_u$ can be written as [11]

$$\boldsymbol{W}_n(\boldsymbol{r}, t) = \int_{\mathcal{B}} \exp\{\mathrm{i}[\boldsymbol{r} \cdot \boldsymbol{k}(b) - \omega(b)t]\}\, u_n(b)\nu(b)\boldsymbol{W}(b)\, \mathrm{d}\mathcal{B} \tag{1}$$

where beam manifold $\mathcal{B}$ is a subset of $\mathcal{B}_u$ with non-vanishing values of function $\boldsymbol{W}' = \nu(b)\boldsymbol{W}(b)$. In such a manner intensities and phases of all eigenwaves are specified by the same complex scalar function u_n satisfying the condition

$$\langle u_m|u_n\rangle \equiv \int_{\mathcal{B}_u} u_m^*(b)u_n(b)\, \mathrm{d}\mathcal{B} = \delta_{mn} \tag{2}$$

where u_m^* is the complex conjugate function to u_m, and δ_{mn} is the Kronecker δ function. Amplitude function $\boldsymbol{W} = \boldsymbol{W}(b)$ must be given in an explicit form for each specific field (see section 3).

There are four key elements defining the properties of these fields: (1) functions (u_n), (2) beam manifold $\mathcal{B}$, (3) beam base, i.e., a set of eigenwaves forming the field and specified by wavevectors $\boldsymbol{k} = \boldsymbol{k}(b)$, angular frequencies $\omega = \omega(b)$ and amplitudes $\boldsymbol{W} = \boldsymbol{W}(b)$, and finally (4) beam state function $\nu = \nu(b)$. By setting these elements in various ways, one can obtain a multitude of specific fields [10–12], among them orthonormal beams satisfying the condition

$$\langle \boldsymbol{W}_m|Q|\boldsymbol{W}_n\rangle \equiv \int_{\sigma_0} \boldsymbol{W}_m^\dagger(\boldsymbol{r}, t) Q \boldsymbol{W}_n(\boldsymbol{r}, t)\, \mathrm{d}\sigma_0 = N_Q\delta_{mn} \tag{3}$$

where σ_0 is either a two- or a three-dimensional manifold, Q is some Hermitian operator, and $\boldsymbol{W}_m^\dagger(\boldsymbol{r}, t)$ is the Hermitian conjugate of $\boldsymbol{W}_m(\boldsymbol{r}, t)$; N_Q is the normalizing constant.

Time-harmonic beams with two-dimensional manifold $\mathcal{B}$ can be written as

$$\boldsymbol{W}_n(\boldsymbol{r}, t) = \exp(-\mathrm{i}\omega t) \int_{\mathcal{B}} \exp[\mathrm{i}\boldsymbol{r} \cdot \boldsymbol{k}(b)] u_n(b)\nu(b)\boldsymbol{W}(b)\, \mathrm{d}\mathcal{B}. \tag{4}$$

They become orthonormal, provided the following conditions are met [11]:

(1) σ_0 is a plane with unit normal $\boldsymbol{q}$, passing through the point $\boldsymbol{r} = 0$.
(2) The tangential component

$$\boldsymbol{t}(b) = \boldsymbol{k}(b) - \boldsymbol{q}[\boldsymbol{q} \cdot \boldsymbol{k}(b)] \tag{5}$$

of $\boldsymbol{k}(b)$ is real for all $b \in \mathcal{B}$, and the mapping $b \mapsto \boldsymbol{t}(b)$ is one–one (injective).

(3) $\mathcal{B} = \mathcal{B}_u$, and the function $\nu(b)$ is given by

$$\nu(b) = \frac{1}{2\pi}\sqrt{\frac{N_Q J(b)}{g(b)\boldsymbol{W}^\dagger(b)Q\boldsymbol{W}(b)}}. \tag{6}$$

Here, $J(b) = D(t^j)/D(\xi^i)$ is the Jacobian determinant of the mapping $b \mapsto \boldsymbol{t}(b)$, calculated in terms of the local coordinate systems $(\xi^i,\ i = 1, 2)$ and $(t^j,\ j = 1, 2)$, and $\mathrm{d}\mathcal{B} = g(b)\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2$.

In the current series of papers, we illustrate the general theory by applying it to time-harmonic fields given by

$$\boldsymbol{W}_j^s(\boldsymbol{r}, t) = \exp(-\mathrm{i}\omega t)\int_0^{2\pi}\mathrm{d}\varphi\int_{\theta_1}^{\theta_2}\exp[\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{k}(\theta,\varphi)]Y_j^s(\theta,\varphi)\nu(\theta,\varphi)\boldsymbol{W}(\theta,\varphi)\sin\theta\,\mathrm{d}\theta. \tag{7}$$

They are defined by the spherical harmonics

$$Y_l^m(\theta,\varphi) = N_{lm}P_l^{|m|}(\cos\theta)\exp(\mathrm{i}m\varphi) \tag{8}$$

where

$$N_{lm} = \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}} \tag{9}$$

and $P_l^m(\cos\theta)$ is the spherical Legendre function [15,16]. For $\boldsymbol{W}_j^s(\boldsymbol{r}, t)$ (7), the beam manifold $\mathcal{B}$ is the spherical zone ($\theta \in [\theta_1, \theta_2]$ and $\varphi \in [0, 2\pi]$) of the unit sphere $\mathcal{B}_u = S^2$, and $\mathrm{d}\mathcal{B} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$.

3. Eigenwave properties

To apply the presented general relations, it is necessary first to calculate parameters of eigenwaves. In this section, we present the corresponding relations for elastic and sound waves.

3.1. Elastic waves

A linear elastic medium is described by the Hooke law [17]

$$\sigma_{ij} = c_{ijlm}\frac{\partial u_m}{\partial x^l} \tag{10}$$

where σ is the stress tensor, $\boldsymbol{u}$ is the displacement vector, $(c_{ijlm}; i, j, l, m = 1, 2, 3)$ are the elastic modules, and summation over repeated indices is carried out from 1 to 3.

3.1.1. Wavevectors and amplitudes. For an eigenwave, the elastodynamics equation [17] $\varrho\partial^2 u_i/\partial t^2 = \partial\sigma_{ij}/\partial x^j$ becomes

$$(\boldsymbol{k}\cdot c\cdot\boldsymbol{k})\boldsymbol{u} = \varrho\omega^2\boldsymbol{u} \tag{11}$$

where ϱ is the medium density, and $\boldsymbol{a}\cdot c\cdot\boldsymbol{b}$ denotes the dyadic with the components $(\boldsymbol{a}\cdot c\cdot\boldsymbol{b})_{im} = a_j c_{ijlm} b_l$. The wavevector surface is defined by the dispersion equation

$$|\boldsymbol{k}\cdot c\cdot\boldsymbol{k} - \varrho\omega^2\mathbf{1}| = 0 \tag{12}$$

which is of sixth order in $\boldsymbol{k}$. Here, $\mathbf{1}$ is the unit dyadic, and $|\Lambda|$ denotes the determinant of a dyadic Λ. If the unit wave normal $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$ is given, equation (12) reduces to the bicubic equation

$$v_\mathrm{p}^6 - v_\mathrm{p}^4\Lambda_t + v_\mathrm{p}^2\overline{\Lambda}_t - |\Lambda| = 0 \tag{13}$$

where $v_{\rm p}$ is the phase velocity ($\boldsymbol{k} = (\omega/v_{\rm p})\hat{\boldsymbol{k}}$), $\Lambda = \hat{\boldsymbol{k}} \cdot c \cdot \hat{\boldsymbol{k}}/\varrho$, $\overline{\Lambda}$ is the adjoint tensor ($\Lambda\overline{\Lambda} = \overline{\Lambda}\Lambda = |\Lambda|\mathbf{1}$), and Λ_t is the trace of Λ.

The eigenwave amplitude $\boldsymbol{u}$ is given by [17]

$$\boldsymbol{u} = \overline{\chi}\boldsymbol{p} \qquad \chi = \Lambda - v_{\rm p}^2\mathbf{1} \tag{14}$$

where $\boldsymbol{p}$ is an arbitrary vector. If χ is a dyad, i.e. $\overline{\chi} = 0$ and $\chi = \boldsymbol{c}_u \otimes \boldsymbol{n}_u$, the amplitude subspace becomes two-dimensional, and $\boldsymbol{u}$ is an arbitrary vector normal to $\boldsymbol{n}_u = \boldsymbol{p}\chi$. To compose a family of orthonormal beams, we shall use the six-dimensional eigenwave amplitude

$$\boldsymbol{W}_0 = \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{f} \end{pmatrix} \qquad \boldsymbol{f} = \sigma\boldsymbol{q} = \mathrm{i}(\boldsymbol{q} \cdot c \cdot \boldsymbol{k})\boldsymbol{u} = \mathrm{i}e_i c_{ijlm} q_j k_l u_m \tag{15}$$

where $\boldsymbol{k}$ and $\boldsymbol{u}$ are specified by equations (12) and (14).

3.1.2. Wavevector surface parametrization by the tangential component $\boldsymbol{t}$ *of* $\boldsymbol{k}$. Substituting $\boldsymbol{k} = \boldsymbol{t} + \xi\boldsymbol{q}$ ($\boldsymbol{t} \cdot \boldsymbol{q} = 0$) in equation (12), we obtain the sixth-order equation in ξ [17]:

$$\left|\xi^2 A + \xi B + C\right| \equiv \sum_{n=1}^{6} a_n \xi^n + |C| = 0 \tag{16}$$

where

$$a_1 = (\overline{C}B)_t \qquad a_2 = (\overline{B}C + \overline{C}A)_t \tag{17}$$

$$a_3 = |B| + (ABC + CBA + AB_tC_t - A_tBC - B_tCA - C_tAB)_t \tag{18}$$

$$a_4 = (\overline{A}C + \overline{B}A)_t \qquad a_5 = (\overline{A}B)_t \qquad a_6 = |A| \tag{19}$$

$$A = \boldsymbol{q} \cdot c \cdot \boldsymbol{q} \qquad C = \boldsymbol{t} \cdot c \cdot \boldsymbol{t} - \varrho\omega^2\mathbf{1} \tag{20}$$

$$B = B_1 + B_2 \qquad B_1 = \boldsymbol{t} \cdot c \cdot \boldsymbol{q} \qquad B_2 = \boldsymbol{q} \cdot c \cdot \boldsymbol{t}. \tag{21}$$

The roots ($\xi_j,\ j = 1, 2, \ldots, 6$) of this equation specify all six wavevectors $\boldsymbol{k}_j = \boldsymbol{t} + \xi_j\boldsymbol{q}$, which have the same given tangential component $\boldsymbol{t}$.

3.1.3. Amplitude orthogonality in a non-dissipative medium. Substituting $\boldsymbol{k}_j = \boldsymbol{t} + \xi_j\boldsymbol{q}$ in equations (11) and (15), we obtain

$$R\boldsymbol{W}_j = \xi_j\boldsymbol{W}_j \qquad \boldsymbol{W}_j = \begin{pmatrix} \boldsymbol{u}_j \\ \boldsymbol{f}_j \end{pmatrix} \tag{22}$$

where

$$R = \begin{pmatrix} -A^{-1}B_2 & -\mathrm{i}A^{-1} \\ \mathrm{i}(B_1A^{-1}B_2 - C) & -B_1A^{-1} \end{pmatrix}. \tag{23}$$

If the dyadics $A,\ B_1,\ B_2$ and C have the properties $A^\dagger = A, \quad C^\dagger = C, \quad B_1^\dagger = B_2$, the block matrix R (23) satisfies the identity

$$R^\dagger = Q_0RQ_0 \qquad Q_0 = \begin{pmatrix} 0 & -\mathrm{i}\mathbf{1} \\ \mathrm{i}\mathbf{1} & 0 \end{pmatrix}. \tag{24}$$

Hence, at $\xi_j \neq \xi_i^*$, equation (22) results in the orthogonality relation $\boldsymbol{W}_i^\dagger Q_0\boldsymbol{W}_j \equiv \mathrm{i}(\boldsymbol{f}_i^* \cdot \boldsymbol{u}_j - \boldsymbol{u}_i^* \cdot \boldsymbol{f}_j) = 0$. This is true in a non-dissipative medium at real values of $\boldsymbol{t}$, since c_{ijlm} has the properties [17] $c_{ijlm} = c_{jilm} = c_{ijml} = c_{lmij} = c_{ijlm}^*$.

As in the case of time-harmonic electromagnetic beams [11], the condition $\langle\boldsymbol{W}_n|Q|\boldsymbol{W}_n\rangle = N_Q$ with $Q = (\omega/4)Q_0$ normalizes the time average elastic beam energy flux N_Q through the plane σ_0:

$$\langle\boldsymbol{W}_n|Q|\boldsymbol{W}_n\rangle = -\tfrac{1}{4}\int_{\sigma_0}(\boldsymbol{v}^* \cdot \boldsymbol{f} + \boldsymbol{v} \cdot \boldsymbol{f}^*)\,\mathrm{d}\sigma_0 = N_Q \tag{25}$$

where the interrelation $\boldsymbol{v} = -\mathrm{i}\omega\boldsymbol{u}$ between the velocity $\boldsymbol{v}$ and the displacement $\boldsymbol{u}$ has been taken into account.

3.2. *Elastic beams in an isotropic medium*

In an isotropic elastic medium, the Hooke law (10) becomes [17]

$$\sigma = \lambda_{\mathrm{L}} \mathrm{div}\, \boldsymbol{u}\, \mathbf{1} + 2\mu_{\mathrm{L}}\gamma \tag{26}$$

where λ_{L} and μ_{L} are the Lamé modules, γ is the deformation tensor with the components

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{27}$$

We shall characterize elastic beams by the time average kinetic and elastic energy densities [17]

$$w_{\mathrm{K}} = \tfrac{1}{4}\varrho\omega^2|\boldsymbol{u}|^2 \qquad w_{\mathrm{E}} = \tfrac{1}{4}\mathrm{Re}\ (\sigma_{ik}\gamma_{ik}^*) \tag{28}$$

and the time average energy flux density vector

$$\boldsymbol{S} = \frac{\omega}{2}\mathrm{Re}\ (\mathrm{i}\sigma^*\boldsymbol{u}). \tag{29}$$

Hence, for an eigenwave with wavevector $\boldsymbol{k} = k\hat{\boldsymbol{k}}$, from equations (11), (15) and (26) it follows that

$$\gamma = \frac{\mathrm{i}k}{2}(\hat{\boldsymbol{k}} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \hat{\boldsymbol{k}}) \tag{30}$$

$$\sigma = \mathrm{i}k[\lambda_{\mathrm{L}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{u})\mathbf{1} + \mu_{\mathrm{L}}(\hat{\boldsymbol{k}} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \hat{\boldsymbol{k}})] \tag{31}$$

$$\boldsymbol{f} = \mathrm{i}k[\lambda_{\mathrm{L}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{u})\boldsymbol{q} + \mu_{\mathrm{L}}(\boldsymbol{u} \cdot \boldsymbol{q})\hat{\boldsymbol{k}} + \mu_{\mathrm{L}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{q})\boldsymbol{u}] \tag{32}$$

$$\Lambda = v_2^2\mathbf{1} + (v_1^2 - v_2^2)\hat{\boldsymbol{k}} \otimes \hat{\boldsymbol{k}} \tag{33}$$

where $v_1 = \sqrt{(\lambda_{\mathrm{L}} + 2\mu_{\mathrm{L}})/\varrho}$ and $v_2 = \sqrt{\mu_{\mathrm{L}}/\varrho}$ are the velocities of longitudinal and transverse elastic waves, respectively.

3.3. *Sound waves*

The general relations presented in section 2 can also be applied to scalar waves. By way of example let us consider sound waves in an ideal liquid. In the linear approximation, the velocity $\boldsymbol{v}$ of fluid particles is assumed to be far less than the sound velocity c_0, and the variations of pressure $p' = p - p_0$ and density $\varrho' = \varrho - \varrho_0$ are assumed to be far less than the equilibrium values p_0 and ϱ_0. Therefore, for an eigenwave, the continuity equation and the Euler equation reduce to [18]

$$\omega\varrho' = \varrho_0\boldsymbol{k} \cdot \boldsymbol{v} \qquad \omega\varrho_0\boldsymbol{v} = p'\boldsymbol{k} \tag{34}$$

where $p' = c_0^2\varrho'$. The dispersion equation $\boldsymbol{k}^2 - \omega^2/c_0^2 = 0$ has two different solutions

$$\boldsymbol{k}_j = \boldsymbol{t} + \xi_j\boldsymbol{q} \quad j = 1, 2 \qquad \xi_{1,2} = \pm\sqrt{\omega^2/c_0^2 - \boldsymbol{t}^2} \tag{35}$$

given the tangential component $\boldsymbol{t}$ of $\boldsymbol{k}$ ($\boldsymbol{t} \cdot \boldsymbol{q} = 0$). From equations (34), (35) we obtain

$$R\boldsymbol{W}_j = \xi_j\boldsymbol{W}_j \qquad \boldsymbol{W}_j = \begin{pmatrix} p'_j \\ \boldsymbol{q} \cdot \boldsymbol{v}_j \end{pmatrix} \quad j = 1, 2 \tag{36}$$

where

$$R = \begin{pmatrix} 0 & \omega\varrho_0 \\ A & 0 \end{pmatrix} \qquad A = \frac{1}{\varrho_0}\left(\frac{\omega}{c_0^2} - \frac{\boldsymbol{t}^2}{\omega}\right) \tag{37}$$

i.e. the amplitude $\boldsymbol{W}_j$ is specified by ξ_j and p'_j as follows:

$$\boldsymbol{W}_j = \begin{pmatrix} p'_j \\ p'_j\xi_j/(\omega\varrho_0) \end{pmatrix}. \tag{38}$$

At real values of $\boldsymbol{t}$, R satisfies the identity

$$R^\dagger = Q_0 R Q_0 \qquad Q_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{39}$$

Equation (36) results in the orthogonality relation [18] $\boldsymbol{W}_1^\dagger Q_0 \boldsymbol{W}_2 = \boldsymbol{q}\cdot(p_1'^*\boldsymbol{v}_2 + p_2'\boldsymbol{v}_1^*) = 0$ at $\xi_1 \neq \xi_2^*$. The condition $\langle \boldsymbol{W}_n|Q|\boldsymbol{W}_n\rangle = N_Q$ with $Q = Q_0/4$ normalizes the time average sound energy flux N_Q through the plane σ_0:

$$\langle \boldsymbol{W}_n|Q|\boldsymbol{W}_n\rangle = \tfrac{1}{4}\int_{\sigma_0} (p'^*\boldsymbol{v} + p'\boldsymbol{v}^*)\cdot\boldsymbol{q}\,\mathrm{d}\sigma_0 = N_Q. \tag{40}$$

We assume below that $\boldsymbol{q} = \boldsymbol{e}_3$.

4. Field parametrization and representation

In an anisotropic medium, substitution of the Rayleigh formula [16]

$$\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi\sum_{l=0}^{+\infty} i^l j_l(kr)\sum_{m=-l}^{l} Y_l^{m*}(\hat{\boldsymbol{k}})Y_l^m(\hat{\boldsymbol{r}}) \tag{41}$$

in equation (7) yields [11]

$$\boldsymbol{W}_j^s(\boldsymbol{r},t) = \mathrm{e}^{-\mathrm{i}\omega t}\sum_{l=0}^{+\infty} i^l \sum_{m=-l}^{l} Y_l^m(\hat{\boldsymbol{r}})\boldsymbol{W}_l^m(r) \tag{42}$$

where

$$\hat{\boldsymbol{k}} = \sin\theta'(\boldsymbol{e}_1\cos\varphi' + \boldsymbol{e}_2\sin\varphi') + \boldsymbol{e}_3\cos\theta' \tag{43}$$

$$\hat{\boldsymbol{r}} = \boldsymbol{r}/r = \sin\gamma(\boldsymbol{e}_1\cos\psi + \boldsymbol{e}_2\sin\psi) + \boldsymbol{e}_3\cos\gamma \tag{44}$$

$Y_l^m(\hat{\boldsymbol{k}})\equiv Y_l^m(\theta',\varphi')$, $Y_l^m(\hat{\boldsymbol{r}})\equiv Y_l^m(\gamma,\psi)$, $j_l(kr)$ is the spherical Bessel function [15, 16], and vector coefficients $\boldsymbol{W}_l^m$ depend only on radius r. In an isotropic medium, this expansion becomes [11]

$$\boldsymbol{W}_j^s(\boldsymbol{r},t) = \mathrm{e}^{-\mathrm{i}\omega t}\sum_{l=0}^{+\infty} i^l j_l(kr)\sum_{m=-l}^{l} Y_l^m(\hat{\boldsymbol{r}})\boldsymbol{W}_l^m. \tag{45}$$

In this case, the field is completely characterized by coordinate-independent vector coefficients $\boldsymbol{W}_l^m$.

To set the function $\boldsymbol{k} = \boldsymbol{k}(\theta,\varphi)$, one can use either the normal $\boldsymbol{k} = k(\theta',\varphi')\hat{\boldsymbol{k}}(\theta',\varphi')$ or the tangential $\boldsymbol{k} = \boldsymbol{t}(\theta',\varphi') + \xi(\theta',\varphi')\boldsymbol{q}$ parametrization (see section 3) with some given functions $\theta' = \theta'(\theta,\varphi)$ and $\varphi' = \varphi'(\theta,\varphi)$. By setting these functions in various ways, one can obtain diverse orthonormal beams and localized fields [10–12]. It is essential that, in the general case, the coordinates θ and φ on $\mathcal{B}$ do not coincide with the spherical coordinates θ' and φ' of $\hat{\boldsymbol{k}}$. We shall restrict our consideration to the fields with $\theta' = \kappa_0\theta$ and $\varphi' = \varphi$, where parameter κ_0 satisfies the condition $0 < \kappa_0 \leqslant 1$. Such fields comprise plane waves with wave normals $\hat{\boldsymbol{k}}$ lying in the same solid angle $\Omega = 2\pi(\cos\kappa_0\theta_1 - \cos\kappa_0\theta_2)$. There are two basically different ways to obtain a family of orthonormal beams [11]. One possibility, applied in this series of papers, is that beams are composed of eigenwaves with different tangential components $\boldsymbol{t}$. To this end, we use two sets of parameters ($\theta_1 = 0$ in both cases): $\theta_2 = \pi/2$ and $\kappa_0 = 1$, and $\theta_2 = \pi$ and $0 < \kappa_0 \leqslant 1/2$, respectively. To specify the amplitude functions, we use the spherical basis vectors

$$\boldsymbol{e}_r(\theta', \varphi) = \sin\theta'(\boldsymbol{e}_1 \cos\varphi + \boldsymbol{e}_2 \sin\varphi) + \boldsymbol{e}_3 \cos\theta' \tag{46}$$

$$\boldsymbol{e}_{\theta'}(\theta', \varphi) = \cos\theta'(\boldsymbol{e}_1 \cos\varphi + \boldsymbol{e}_2 \sin\varphi) - \boldsymbol{e}_3 \sin\theta' \tag{47}$$

$$\boldsymbol{e}_\varphi(\varphi) = -\boldsymbol{e}_1 \sin\varphi + \boldsymbol{e}_2 \cos\varphi. \tag{48}$$

To find the coefficients $\boldsymbol{W}_l^m$, we shall extensively use functions $I_j^{sm}[f]$ defined as follows [11]:

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi/2} \exp\{\mathrm{i}[k\boldsymbol{r} \cdot \boldsymbol{e}_r(\theta, \varphi) + n\varphi]\}\, Y_j^s(\theta, \varphi) f(\theta) \sin\theta \,\mathrm{d}\theta = \exp[\mathrm{i}(s+n)\psi] I_j^{ss+n}[f](r, \gamma) \tag{49}$$

$$I_j^{sm}[f] = I_j^{sm}[f](r, \gamma) = \sum_{l=|m|}^{+\infty} i^l j_l(kr) P_l^{|m|}(\cos\gamma) \mathcal{P}_{jl}^{sm}[f] \tag{50}$$

$$\mathcal{P}_{jl}^{sm}[f] = 8\pi^2 N_{js} N_{lm}^2 \int_0^{\pi/2} P_j^{|s|}(\cos\theta) P_l^{|m|}(\cos\theta) f(\theta) \sin\theta \,\mathrm{d}\theta \tag{51}$$

where $f = f(\theta)$ is a scalar function of the polar angle θ, and n is an integer. Function $I_j^{sm}[f]$ at fixed r and γ as well as coefficient $\mathcal{P}_{jl}^{sm}[f]$ are functionals regarding f. For any given f, $I_j^{sm}[f]$ is a function of r and γ, whereas $\mathcal{P}_{jl}^{sm}[f]$ is a constant. We omit the arguments (r, γ) where appropriate. The real and imaginary parts of $I_j^{sm}[f]$ can be separated as

$$I_j^{sm}[f] = i^{|m|}(J_{j0}^{sm}[f] + i J_{j1}^{sm}[f]) \tag{52}$$

where

$$J_{jp}^{sm}[f] = J_{jp}^{sm}[f](r, \gamma) = \sum_{\nu=0}^{+\infty} (-1)^\nu j_{|m|+2\nu+p}(kr) P_{|m|+2\nu+p}^{|m|}(\cos\gamma) \mathcal{P}_{j|m|+2\nu+p}^{s|m|}[f]. \tag{53}$$

Additional information on these functions can be found in [11].

5. Conclusion

The relations presented in this paper provide means to extend the formalism proposed in [11] to elastic beams in isotropic and anisotropic mediums and sound beams in an ideal liquid. This formalism makes it possible to obtain exact solutions of linear field equations, which describe families of orthonormal beams and various types of localized fields. Owing to the orthonormality conditions, the families of orthonormal beams constitute convenient functional bases for complex elastic and sound fields and can be applied for modeling the beams now in use. In the subsequent three papers of the current series, it will be shown that the localized elastic and sound fields also possess interesting properties. In particular, they have a very small (about several wavelengths) core region with maximum intensity of field oscillations and unique energy transport.

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